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CQG Algebras: A Direct Algebraic Approach to Compact Quantum Groups

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Abstract. The purely algebraic notion of CQG algebra (algebra of functions on a compact quantum group) is defined. In a straightforward algebraic manner, the Peter–Weyl theorem for CQG algebras and the existence of a unique positive definite Haar functional on any CQG algebra are established. It is shown that a CQG algebra can be naturally completed to a C^*-algebra. The relations between our approach and several other approaches to compact quantum groups are discussed.

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0. Introduction

Compact quantum groups are fairly well understood on the one hand for special cases such as $SU_q(2)$ (cf. the early paper [13]) and, more generally, quantum analogues of the classical compact Lie groups and beyond (cf. [11]) and, on the other hand, in a general theory started by Woronowicz [16–18] for compact matrix quantum groups. A crucial aspect of Woronowicz's general theory is the existence theorem for a positive Haar functional. Some C^*-algebra theory is used in the demonstration of that theorem and, actually, a C^*-algebra is already present in Woronowicz's definition of a compact matrix quantum group. This is in contrast with the special cases where a Hopf $*$-algebra is presented as an algebra by generators and relations. The demonstration of a C^*-completion for such explicit algebras can be quite cumbersome and is actually not necessary for many applications where one is only interested in algebraic aspects. Thus, in special cases, one usually develops the theory in an ad-hoc algebraic manner, and in this way one arrives at results fitting into Woronowicz's general theory without actually invoking his theorems. By the way, also in the general case, the main results of Woronowicz's

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theory (Schur orthogonality relations, Peter–Weyl theorem) can be formulated in a meaningful way on the algebraic level.

In this Letter, we propose a purely algebraic approach to general compact quantum groups. After some preliminaries in Section 1, we define, in Section 2, a CQG algebra (associated with a compact quantum group) as a Hopf $*$-algebra which is the linear span of the matrix elements of its finite-dimensional unitary corepresentations. A CMQG algebra (associated with a compact matrix quantum group) is then a finitely generated CQG algebra. These definitions and the subsequent development of the theory (in Section 3) do not involve C^*-algebras. All main results of [16], as far as they are on the Hopf $*$-algebra level, are thus proved in an algebraic way. In particular, the existence of a unique (not a-priori positive) Haar functional on a CQG algebra is immediate, and its positivity and faithfulness (on the CQG algebra) is one of the results of Section 3. We show in Section 4 that a CQG algebra has a natural C^*-completion, by which we make contact with [16]. The Letter concludes (in Section 5) with a comparison of various approaches to compact quantum groups which have appeared in the literature. Particular mention should be made here of the paper [5] by Effros and Ruan, who earlier introduced the same algebras as our CQG algebras, but called them differently and also developed the theory in a different direction. We also mention the paper [2] (see also [6]), in which, among other things, a notion of so-called preferred deformation of the algebra of representative functions on a compact connected Lie group is defined and a number of its properties are studied. Although the techniques used in [2] are quite different, some of the results are similar in spirit to ours.

The results presented here are also part of the PhD thesis of Dijkhuizen [4], while a more tutorial presentation will appear in lecture notes by Koornwinder [8].

1. Preliminaries

All vector spaces are taken over the field of complex numbers $\mathbb{C}$. All tensor products of vector spaces are algebraic unless explicitly mentioned otherwise. We canonically identify the tensor products $V \otimes \mathbb{C}$ and $\mathbb{C} \otimes V$ with V, for any vector space V. For more information about the material treated in this section, see [12, 4, 8].

In a Hopf algebra A, we write $\Delta\colon A \to A \otimes A$ for the comultiplication, $\varepsilon\colon A \to \mathbb{C}$ for the counit, and $S\colon A \to A$ for the antipode. We recall the symbolic notation for Δ and its iterates:

$$\Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)},$$

$$(\Delta \otimes \mathrm{id}) \circ \Delta(a) = (\mathrm{id} \otimes \Delta) \circ \Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)} \otimes a_{(3)}. \tag{1.1}$$

A *Hopf $*$-algebra* is a Hopf algebra A endowed with a conjugate linear involutive mapping $*\colon A \to A$ such that A as an algebra is a $*$-algebra and such that Δ and ε are $*$homomorphisms. It then follows that the antipode S is bijective and satisfies $S \circ * \circ S \circ * = \mathrm{id}$.

Let A be a Hopf algebra. A *corepresentation* of A in a vector space V is a linear mapping $\pi: V \to V \otimes A$ such that

$$(\pi \otimes \mathrm{id}) \circ \pi = (\mathrm{id} \otimes \Delta) \circ \pi, \qquad (\mathrm{id} \otimes \varepsilon) \circ \pi = \mathrm{id}. \tag{1.2}$$

We shall sometimes use the following symbolic notation for π and its iterates:

$$\pi(v) = \sum_{(v)} v_{(1)} \otimes v_{(2)},$$
$$(\pi \otimes \mathrm{id}) \circ \pi(v) = (\mathrm{id} \otimes \Delta) \circ \pi(v) = \sum_{(v)} v_{(1)} \otimes v_{(2)} \otimes v_{(3)}. \tag{1.3}$$

Here $v \in V$, the $v_{(1)}$ are in V, and the $v_{(2)}, v_{(3)}$ in A. If the corepresentation space V is finite-dimensional and $\{v_i\}$ is a basis of V, then we write $\pi(v_j) = \Sigma_i v_i \otimes \pi_{ij}$, where the π_{ij} are elements of A. Then $\pi = (\pi_{ij})$ is a matrix corepresentation of A:

$$\Delta(\pi_{ij}) = \sum_k \pi_{ik} \otimes \pi_{kj}, \qquad \varepsilon(\pi_{ij}) = \delta_{ij}. \tag{1.4}$$

Given two corepresentations π in V and ρ in W, a linear mapping $\varphi: V \to W$ is called an *intertwining operator* if $\rho \circ \varphi = (\varphi \otimes \mathrm{id}) \circ \pi$.

Given a corepresentation π of A in a finite-dimensional vector space V, the *contragredient corepresentation* π' of π is the corepresentation of A in the linear dual V' defined by

$$(v \otimes \mathrm{id}) \circ \pi'(v') = (v' \otimes S) \circ \pi(v), \quad v' \in V', v \in V. \tag{1.5}$$

If we write $\pi = (\pi_{ij})$ with respect to a basis $\{v_i\}$ of V and $\pi' = (\pi'_{ij})$ with respect to the dual basis $\{v^i\}$ of V', then

$$\pi'_{ij} = S(\pi_{ji}). \tag{1.6}$$

Given two corepresentations π, ρ in finite-dimensional vector spaces V, W, respectively, their *tensor product* $\pi \boxtimes \rho$ is the corepresentation of A in the vector space $V \otimes W$ defined in symbolic notation by

$$\pi \boxtimes \rho(v \otimes w) = \sum_{(v),(w)} v_{(1)} \otimes w_{(1)} \otimes v_{(2)} w_{(2)}. \tag{1.7}$$

If we write $\pi = (\pi_{ij})$ and $\rho = (\rho_{kl})$ with respect to a basis $\{v_i\}$ of V and $\{w_k\}$ of W and if we write $\pi \boxtimes \rho = ((\pi \boxtimes \rho)_{ik,jl})$ with respect to the basis $\{v_i \otimes w_k\}$ of $V \otimes W$, then $(\pi \boxtimes \rho)_{ik,jl} = \pi_{ij} \rho_{kl}$.

Suppose A is a Hopf $*$-algebra. Let V be a vector space endowed with an inner product. A corepresentation π of A in V is called *unitary* if

$$\sum_{(v)} \langle v_{(1)}, w \rangle S(v_{(2)}) = \sum_{(w)} \langle v, w_{(1)} \rangle w_{(2)}^* \quad \forall v, w \in V. \tag{1.8}$$

Suppose V is finite-dimensional and $\pi = (\pi_{ij})$ with respect to an orthonormal basis $\{v_i\}$ of V. Then π is unitary if and only if the following equivalent conditions are satisfied:

$$S(\pi_{ij}) = \pi_{ji}^* \Leftrightarrow \sum_k \pi_{ki}^* \pi_{kj} = \delta_{ij} 1 \Leftrightarrow \sum_k \pi_{ik} \pi_{jk}^* = \delta_{ij} 1. \tag{1.9}$$

A corepresentation π in a vector space V is called *unitarizable* if there exists an inner product on V such that π is unitary with respect to this inner product.

The usual notions from representation theory such as direct sums, invariant subspaces, irreducibility, complete reducibility, etc., all have an obvious meaning in corepresentation theory. Note that a unitary corepresentation is always completely reducible, since the orthogonal complement of an invariant subspace is again invariant.

2. CQG Algebras

THEOREM 2.1. *Let A be a Hopf algebra. Let Σ denote the set of equivalence classes of finite-dimensional irreducible corepresentations of A. For each $\alpha \in \Sigma$, let the matrix corepresentation $\pi^\alpha = (\pi^\alpha_{ij})$ be a representative of the class α and let $A_\alpha \subset A$ denote the span of its matrix coefficients. Then $\sum_{\alpha\in\Sigma} A_\alpha$ is a direct sum and the π^α_{ij} are linearly independent.*

This theorem is rather standard. In 2.1.3, 1.1.33, 1.1.54, and 1.1.16 of [4], it is pointed out that the theorem is already valid in the case of finite-dimensional irreducible corepresentations of a coalgebra, and that, by duality, the proof can be reduced to a similar result for algebra representations, see for instance [3] Section 13, No. 3, pp. 154–155. A self-contained proof not referring to algebra representations but assuming that S is invertible, is given in [8], Proposition 1.28.

From now on, we will work with a Hopf $*$-algebra A, and Σ will denote the set of equivalence classes of finite-dimensional irreducible *unitary* corepresentations of A. For $\alpha \in \Sigma$, let π^α and A_α be as in the above theorem, but the matrix corepresentation (π^α_{ij}) is now supposed to be unitary.

DEFINITION 2.2. A *CQG algebra* is a Hopf $*$-algebra which is spanned by the coefficients of its finite-dimensional unitary (irreducible) corepresentations. We then say that A is the Hopf $*$-algebra associated with a compact quantum group. The decomposition $A = \sum_{\alpha\in\Sigma} A_\alpha$ is called the *Peter–Weyl decomposition* of A.

PROPOSITION 2.3. *Let A be a CQG algebra. Then:*

(i) *Every finite-dimensional irreducible corepresentation of A is equivalent to some π^α $(\alpha \in \Sigma)$.*
(ii) *Every finite-dimensional irreducible corepresentation of A is unitarizable.*
(iii) *If π is a finite-dimensional unitarizable matrix corepresentation of A then so is its contragredient π'.*

Assertion (i) is an immediate consequence of Theorem 2.1 and Definition 2.2. Assertion (ii) follows from (i). For (iii), observe that a finite-dimensional unitarizable corepresentation π is completely reducible, hence so is π'.

PROPOSITION 2.4. *For a Hopf $*$-algebra A the following conditions are equivalent:*

(i) *A is a finitely generated CQG algebra.*
(ii) *There is a finite-dimensional unitary corepresentation of A whose matrix coefficients generate A as an algebra.*
(iii) *There is a finite-dimensional corepresentation π of A such that both π and π' are unitarizable and such that A is generated as an algebra by the matrix coefficients of π and π'.*
(iv) *There is a finite subset $\{\alpha_1, \ldots, \alpha_n\} \subset \Sigma$ such that the matrix coefficients of the π^{α_i} $(1 \leqslant i \leqslant n)$ generate A as an algebra.*

Assume (i) and pick a finite set of generators of A. By definition of CQG algebra, each of the generators is a linear combination of matrix coefficients of finite-dimensional unitary corepresentations of A. Taking the direct sum of all the corepresentations involved, we obtain a finite-dimensional unitary corepresentation whose matrix coefficients generate A. This proves (i) $\Rightarrow$ (ii). The implication (ii) $\Rightarrow$ (iii) follows from Proposition 2.3(iii). The implications (iii) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (i) are immediate. This concludes the proof.

DEFINITION 2.5. A CQG algebra is called a *CMQG algebra* if it satisfies the equivalent conditions of [2.4]. A CMQG algebra is said to be associated with a compact matrix quantum group.

It can be easily shown (cf. [4] pp. 57–60) that the category of CQG algebras is closed under taking inductive limits (for a definition of inductive limit see also [10] p. 67). The standard fact that any compact group can be written as a projective limit of compact Lie groups, generalizes to the statement that any CQG algebra is the inductive limit of CMQG algebras. Conversely, given a family of CQG algebras $(A_\lambda)_{\lambda \in \Lambda}$, the tensor products of finite subfamilies of $(A_\lambda)_{\lambda \in \Lambda}$ naturally form an inductive family whose limit generally is a nonfinitely generated CQG algebra. In this way, one can construct examples of nonfinitely generated CQG algebras starting from an infinite family of CMQG algebras.

3. The Haar Functional

We now discuss the concept of Haar functional and its relation to CQG algebras.

DEFINITION 3.1. Let A be a Hopf $*$-algebra. A *Haar functional* on A is a linear functional $h\colon A \to \mathbb{C}$ which satisfies $h(1) = 1$, and is such that

$$(h \otimes \mathrm{id}) \circ \Delta(a) = h(a)1 = (\mathrm{id} \otimes h) \circ \Delta(a), \quad a \in A. \tag{3.1}$$

The invariance (3.1) of a Haar functional $h\colon A \to \mathbb{C}$ with respect to comultiplication can also be written as

$$\sum_{(a)} h(a_{(1)})a_{(2)} = h(a)1 = \sum_{(a)} h(a_{(2)})a_{(1)}, \quad a \in A. \tag{3.2}$$

A Haar functional $h: A \to \mathbb{C}$ is called *positive* if $h(a^*a) \geqslant 0$ for all $a \in A$, and it is called *positive definite* or *faithful* if $h(a^*a) > 0$ for all nonzero $a \in A$. It can be easily proved that, if a Haar functional h satisfies $h(a^*a) \in \mathbb{R}$ for all $a \in A$, then $h(a^*) = \overline{h(a)}$ for all $a \in A$.

Let now A be a CQG algebra. Denote by 1 the unique $\alpha \in \Sigma$ such that π^α is equivalent to the one-dimensional unitary matrix corepresentation (1). Also, for each $\alpha \in \Sigma$, let α' be the unique $\beta \in \Sigma$ such that π^β is equivalent to $(\pi^\alpha)'$, cf. Proposition 2.3(iii).

We now define a linear form $h: A \to \mathbb{C}$ by setting

$$h(a) = \begin{cases} 0, & \text{if } a \in A_\alpha, \alpha \neq 1, \\ 1, & \text{if } a = 1. \end{cases} \tag{3.3}$$

PROPOSITION 3.2. *Let A be a CQG algebra and let $h: A \to \mathbb{C}$ be the linear form defined in* (3.3). *Then h is a Haar functional on A and satisfies $h(S(a)) = h(a)$ and $h(a^*) = \overline{h(a)}$ for all $a \in A$. Any linear functional $h': A \to \mathbb{C}$ such that $h'(1) = 1$ and $(h' \otimes \mathrm{id}) \circ \Delta(a) = h'(a)1$ for all $a \in A$ is equal to h.*

The *proof* is completely elementary.

Let G be a compact group and let A be the corresponding CQG algebra of representative functions on G. If $\mathrm{d}x$ denotes the Haar measure on G, then the Haar functional on A is given by

$$h(a) = \int_G a(x)\,\mathrm{d}x$$

and (3.1) expresses the invariance of the Haar measure with respect to group multiplication. It is well known that the Haar measure is a positive measure and that the support of $\mathrm{d}x$ is equal to G, in other words, the Haar functional h is positive definite. We are going to prove the same result for a general CQG algebra. We first prove an important lemma:

LEMMA 3.3. *Let A be a Hopf algebra and suppose $h: A \to \mathbb{C}$ is a linear form satisfying* (3.1) *and such that $h(1) = 1$. Let ρ and σ be matrix corepresentations of A. Then*

$$\sum_l h(\sigma_{ij}S(\rho_{kl}))\rho_{lm} = \sum_l \sigma_{il}h(\sigma_{lj}S(\rho_{km})), \tag{3.4}$$

$$\sum_l h(S(\rho_{ij})\sigma_{kl})\sigma_{lm} = \sum_l \rho_{il}h(S(\rho_{lj})\sigma_{km}). \tag{3.5}$$

With the notation

$$A_{il}^{(j,k)} := h(\sigma_{ij}S(\rho_{kl})), \qquad B_{il}^{(j,k)} := h(S(\rho_{ij})\sigma_{kl}), \tag{3.6}$$

the identities (3.4) *and* (3.5) *can be rewritten as*

$$A^{(j,k)}\rho = \sigma A^{(j,k)}, \qquad B^{(j,k)}\sigma = \rho B^{(j,k)}. \tag{3.7}$$

Thus, $A^{(j,k)}$ is an intertwining operator for ρ and σ, and $B^{(j,k)}$ is an intertwining operator for σ and ρ.

For the proof of (3.4) we write

$$h(\sigma_{ij}S(\rho_{kl}))1 = (\mathrm{id} \otimes h)(\Delta(\sigma_{ij}S(\rho_{kl})))$$
$$= \sum_{p,n} (\mathrm{id} \otimes h)((\sigma_{ip} \otimes \sigma_{pj})(S(\rho_{nl}) \otimes S(\rho_{kn}))$$
$$= \sum_{p,n} h(\sigma_{pj}S(\rho_{kn}))\sigma_{ip}S(\rho_{nl}).$$

Substituting this equality in the left-hand side of Equation (3.4) and then using that $\sum_l S(\rho_{nl})\rho_{lm} = \delta_{nm}1$, we arrive at (3.4). Similarly, (3.5) is obtained from a substitution of

$$h(S(\rho_{lj})\sigma_{km})1 = (h \otimes \mathrm{id})(\Delta(S(\rho_{lj})\sigma_{km}))$$

in the right-hand side of (3.5).

PROPOSITION 3.4. *Let A be a Hopf algebra and suppose $h\colon A \to \mathbb{C}$ is a linear form satisfying* (3.1) *and such that $h(1) = 1$. If ρ and σ are nonequivalent irreducible matrix corepresentations of A, then*

$$h(\sigma_{kl}S(\rho_{ij})) = 0, \qquad h(S(\rho_{kl})\sigma_{ij}) = 0.$$

We use the notation of Lemma 3.3. By the Schur lemma for corepresentations, (3.7) yields that $A_{il}^{(j,k)} = 0$ and $B_{il}^{(j,k)} = 0$. Hence, by (3.6), the assertion.

If (π, V) is a finite-dimensional corepresentation of a Hopf algebra A, then the corepresentation space V'' of the double contragredient corepresentation π'' of π can be naturally identified with V.

PROPOSITION 3.5. *Let A be a Hopf algebra with invertible antipode S and suppose $h\colon A \to \mathbb{C}$ is a linear form satisfying* (3.1) *and such that $h(1) = 1$ and $h(S(a)) = h(a)$. Let ρ be an irreducible matrix corepresentation of A. Then ρ is equivalent to its double contragredient ρ''. Let F be any invertible operator intertwining ρ and ρ''. Then* $\mathrm{tr}(F) \neq 0$ *and* $\mathrm{tr}(F^{-1}) \neq 0$ *and*

$$h(\rho_{kl}S(\rho_{ij})) = \delta_{kj}\frac{F_{il}}{\mathrm{tr}(F)}, \tag{3.8}$$

$$h(S(\rho_{kl})\rho_{ij}) = \delta_{kj}\frac{(F^{-1})_{il}}{\mathrm{tr}(F^{-1})}. \tag{3.9}$$

Putting $\sigma = \rho''$ in [3.3] and using the fact that $h(S(a)) = h(a)$, we deduce from (3.4) and (3.5) that

$$\sum_l h(\rho_{kl}S(\rho_{ij}))\rho_{lm} = \sum_l S^2(\rho_{il})h(\rho_{km}S(\rho_{lj})),$$
$$\sum_l h(S(\rho_{kl})\rho_{ij})S^2(\rho_{lm}) = \sum_l \rho_{il}h(S(\rho_{km})\rho_{lj}).$$

With the notation

$$\tilde{A}_{il}^{(j,k)} = h(\rho_{kl}S(\rho_{ij})), \qquad \tilde{B}_{il}^{(j,k)} = h(S(\rho_{kl})\rho_{ij}), \tag{3.10}$$

we have

$$\tilde{A}^{(j,k)}\rho = \rho'' A^{(j,k)} \quad \text{and} \quad \tilde{B}^{(j,k)}\rho'' = \rho \tilde{B}^{(j,k)}.$$

On the other hand, if we apply Lemma 3.3 to ρ and $\sigma = \rho$, we obtain operators $A^{(j,k)}$ and $B^{(j,k)}$ intertwining ρ with itself. It follows from (3.6) that

$$\tilde{A}_{il}^{(j,k)} = A_{kj}^{(l,i)}, \qquad \tilde{B}_{il}^{(j,k)} = B_{kj}^{(l,i)}. \tag{3.11}$$

By the Schur lemma, there are complex numbers α_{jk} and β_{jk} such that

$$A_{il}^{(j,k)} = \alpha_{jk}\delta_{il}, \qquad B_{il}^{(j,k)} = \beta_{jk}\delta_{il}, \tag{3.12}$$

since ρ is irreducible. If we sum over $i = l$ in (3.10) we get

$$\operatorname{tr} \tilde{A}^{(j,k)} = \delta_{jk}, \qquad \operatorname{tr} \tilde{B}^{(j,k)} = \delta_{jk}. \tag{3.13}$$

Hence, there exists a nonzero intertwining operator F for ρ and ρ''. Since S is invertible, ρ and ρ'' are both irreducible and therefore F is invertible. So ρ and ρ'' are equivalent corepresentations. Since an intertwining operator between equivalent irreducible corepresentations is uniquely determined up to a scalar factor, we conclude from (3.13) that $\operatorname{tr}(F) \neq 0$ and $\operatorname{tr}(F^{-1}) \neq 0$. Moreover, there exist complex numbers $\tilde{\alpha}_{jk}$ and $\tilde{\beta}_{jk}$ such that

$$\tilde{A}_{il}^{(j,k)} = \tilde{\alpha}_{jk} F_{il}, \qquad \tilde{B}_{il}^{(j,k)} = \beta_{jk}(F^{-1})_{il}. \tag{3.14}$$

Combination with (3.13) yields that

$$\tilde{\alpha}_{jk} \operatorname{tr}(F) = \delta_{kj}, \qquad \tilde{\beta}_{jk} \operatorname{tr}(F^{-1}) = \delta_{kj}. \tag{3.15}$$

The identities (3.8) and (3.9) follow from (3.10), (3.14) and (3.15). This concludes the proof.

Let V be a finite-dimensional vector space with inner product $\langle\, ,\rangle$. We recall that a linear mapping $T\colon V \to V$ is called *positive definite* if T is self-adjoint, i.e. $T = T^*$, and if $\langle Tx, x\rangle > 0$ for all $x \neq 0$. A matrix (T_{ij}) is positive definite (with respect to the canonical inner product on $\mathbb{C}^n$) if and only if $T_{ij} = \overline{T_{ji}}$ and $\sum_{i,j} x_i \bar{x}_j T_{ji} > 0$ for any n-tuple $(x_1, \ldots, x_n) \neq 0$. If T is any invertible matrix then ${}^t T \bar{T}$ is positive definite, where $\bar{T} = (\overline{T_{ij}})$.

PROPOSITION 3.6. *Let A be a CQG algebra and let ρ be a finite-dimensional irreducible unitary corepresentation of A in an inner product space V. Let F be an invertible operator intertwining ρ and ρ''. Then F is a constant multiple of a positive definite operator on V. It can be uniquely normalized such that* $\operatorname{tr}(F) = \operatorname{tr}(F^{-1}) > 0$.

Let us fix an orthonormal basis of V. Then ρ and ρ'' can be viewed as matrix corepresentations. By Proposition 2.3(iii), there is a unitary matrix corepresentation σ which is equivalent to ρ'. So there is an invertible complex matrix T such that $\sigma T = T\rho'$. By (1.6) and the unitarity of σ and ρ, we have the identities

$$\sigma'_{ij} = \sigma^*_{ij} = S(\sigma_{ji}), \qquad \rho'_{ij} = \rho^*_{ij}, \qquad \rho''_{ij} = S(\rho'_{ji}).$$

It now follows from $\sigma T = T\rho'$ that $\sigma'\bar{T} = \bar{T}\rho$ and ${}^tT\sigma' = \rho''{}^tT$. Hence, ${}^tT\bar{T}\rho = \rho''{}^tT\bar{T}$. In other words, ${}^tT\bar{T}$ intertwines ρ and ρ''. Therefore, F is a constant multiple of ${}^tT\bar{T}$. Since ${}^tT\bar{T}$ is a positive definite matrix, the first assertion follows. The second one is trivial.

THEOREM 3.7. *Let A be a CQG algebra. Then the Haar functional $h: A \to \mathbb{C}$ is positive definite.*

It follows from Propositions 3.4, 3.5, and 3.6 that there exist positive definite matrices G^α such that

$$h((\pi^\alpha_{kl})^*\pi^\beta_{ij}) = \delta_{\alpha\beta}\delta_{lj}G^\alpha_{ik}, \quad \alpha, \beta \in \Sigma.$$

Let $a = \sum_{\alpha,k,l} c^\alpha_{kl}\pi^\alpha_{kl}$ be an arbitrary element of A. Then

$$h(a^*a) = \sum_{\alpha,l}\sum_{i,k} \overline{c^\alpha_{kl}}c^\alpha_{il}G^\alpha_{ik} \geqslant 0,$$

since G^α is positive for every $\alpha \in \Sigma$. Suppose $h(a^*a) = 0$. Then $\sum_{ik} \overline{c^\alpha_{kl}}c^\alpha_{il}G^\alpha_{ik} = 0$ for all α and k. By positive definiteness of the G^α this implies that all coefficients c^α_{kl} are 0, whence $a = 0$.

Remark 3.8. The way we have proved Theorem 3.7 is quite analogous to the proof of Proposition 3.5 in [17].

PROPOSITION 3.9. *Let A be a Hopf $*$-algebra on which there exists a positive definite Haar functional. Then any finite-dimensional corepresentation π of A is unitarizable and therefore completely reducible. In particular, the conclusion holds if A is a CQG algebra.*

Let us denote the corepresentation space of the finite-dimensional corepresentation π by V and let $\langle\,,\rangle$ be any inner product on V. We define a new inner product $\langle\,,\rangle_h$ on V by putting

$$\langle v, w\rangle_h = \sum_{(v),(w)} \langle v_{(1)}, w_{(1)}\rangle h(w^*_{(2)}v_{(2)}).$$

Indeed, it is clear that $\langle\,,\rangle_h$ is a Hermitian form. Let (v_i) be an orthonormal basis of V with respect to the inner product $\langle\,,\rangle$ and let us write π_{ij} for the matrix coefficients of π with respect to this basis. Then

$$\langle v_i, v_j\rangle_h = \sum_{k,l} \langle v_k, v_l\rangle h(\pi^*_{lj}\pi_{ki}) = \sum_k h(\pi^*_{kj}\pi_{ki}).$$

Hence

$$\left\langle \sum_i c_i v_i, \sum_j c_j v_j \right\rangle_h = \sum_k h\left(\left(\sum_j c_j\pi_{kj}\right)^*\left(\sum_i c_i\pi_{ki}\right)\right) \geqslant 0,$$

and if the left-hand side equals 0, then $\sum_i c_i\pi_{ki} = 0$ for all k by the positive definiteness of h. Hence, $c_k = \varepsilon(\sum_i c_i\pi_{ki}) = 0$ for all k, which proves that $\langle\,,\rangle_h$ is an inner product on V. Using (3.2), one deduces

$$\sum_{(v),(w)} \langle v_{(1)}, w_{(1)} \rangle_h w^*_{(2)} v_{(2)}$$

$$= \sum_{(v),(w)} \langle v_{(1)}, w_{(1)} \rangle h(w^*_{(2)} v_{(2)}) w^*_{(3)} v_{(3)}$$

$$= \sum_{(v),(w)} \langle v_{(1)}, w_{(1)} \rangle h(w^*_{(2)} v_{(2)}) 1 = \langle v, w \rangle_h 1,$$

in other words, the corepresentation π is unitary with respect to the inner product $\langle\,,\rangle_h$.

THEOREM 3.10. *Let A be a Hopf $*$-algebra. Then there exists a positive definite Haar functional on A if and only if A is a CQG algebra.*

The implication $\Leftarrow$ follows from Theorem 3.7. Conversely, suppose that there exists a positive definite Haar functional on the Hopf $*$-algebra A. By Proposition 3.9 any finite-dimensional corepresentation of A is unitarizable. We next claim that every element $a \in A$ occurs as a matrix coefficient of some finite-dimensional (hence, unitarizable) corepresentation of A. Indeed, by the Fundamental Theorem on Coalgebras (cf. [12], Th. 2.2.1, p. 46) there is a finite-dimensional subcoalgebra C of A containing a. Let us denote the restriction of Δ to C by π. Then clearly π is a corepresentation of A in the finite-dimensional vector space C. To prove our claim, it suffices to exhibit an element $c \in C$ and a linear form c' on C such that $(c' \otimes \mathrm{id}) \circ \pi(c) = a$. We take $c = a$ and $c' = \varepsilon_{|C}$. It is trivial to check that this works. This concludes the proof.

4. C^*-Algebra Completion

We shall now show that any CQG algebra can be naturally completed to a unital C^*-algebra. Let us recall that a $*$-representation of a $*$-algebra A in a Hilbert space $\mathscr{H}$ is a $*$-algebra homomorphism of A into the algebra $\mathscr{L}(\mathscr{H})$ of bounded operators on $\mathscr{H}$.

Let A be a unital $*$-algebra. A seminorm p on A is called a *C^*-seminorm* if $p(ab) \leqslant p(a)p(b)$ and $p(a^*a) = p(a)^2$. It then automatically follows that $p(a^*) = p(a)$. In addition, if $p \neq 0$ then $p(1) = 1$. A C^*-seminorm p on A is called a *C^*-norm* if $p(a) = 0$ implies $a = 0$. Given a C^*-norm p on A, the completion of A with respect to p naturally is a unital C^*-algebra such that the canonical injection of A into its completion is a $*$-algebra homomorphism. If $\pi\colon A \to B$ is a $*$-algebra homomorphism of A into a C^*-algebra B, then the mapping $a \mapsto \|\pi(a)\|$ is a C^*-seminorm on A. In particular, every $*$-representation of A gives rise to a C^*-seminorm.

LEMMA 4.1. *Let A be a $*$-algebra with a C^*-norm p. For any $a \in A$, there exists an irreducible $*$-representation $\pi\colon A \to \mathscr{L}(\mathscr{H})$ of A in some Hilbert space $\mathscr{H}$ such that $\|\pi(a)\| = p(a)$.*

The result will follow from the corresponding statement for the C^*-algebra completion $\tilde{A}$ of A. For a proof in that case see, for instance, [1] (Corollary to Theorem 1.7.2, p. 34).

LEMMA 4.2. *Let A be a Hopf $*$-algebra. Let π be an algebra homomorphism of A into the algebra of linear operators on some pre-Hilbert space V such that $(\pi(a)v, w) = (v, \pi(a^*)w)$ for all $a \in A$ and all $v, w \in V$. Let $\rho = (\rho_{ij})$ be a unitary matrix corepresentation of A. Then $\pi(\rho_{ij})$ is a bounded linear operator on V of norm $\leqslant 1$ for all i, j. If A is a CQG algebra then π can be uniquely extended to a $*$-representation of A in the Hilbert space completion of V.*

Since ρ is unitary, we have $\sum_k \rho_{kj}^* \rho_{kj} = 1$. Hence, for all $v \in V$,

$$\| v \|^2 = (v, v) = \sum_k (\pi(\rho_{kj}^* \rho_{kj})v, v) = \sum_k (\pi(\rho_{kj})v, \pi(\rho_{kj})v) \geqslant \| \pi(\rho_{ij})v \|^2.$$

LEMMA 4.3. *Let A be a Hopf $*$-algebra and let p be a C^*-seminorm on A. If $\rho = (\rho_{ij})$ is a unitary matrix corepresentation of A, then $p(\rho_{ij}) \leqslant 1$ for all i, j.*

The subset $N = \{a \in A \mid p(a) = 0\}$ is a two-sided $*$-ideal in A and the quotient A/N naturally is a $*$-algebra. Let $\varphi: A \to A/N$ denote the canonical surjection. Then we can put $\bar{p}(\varphi(a)) = p(a)$, and $\bar{p}$ clearly is a C^*-norm on A/N. Fix i, j. By Lemma 4.1 there is a $*$-representation $\bar{\pi}$ of A/N in some Hilbert space $\mathscr{H}$ such that $\| \bar{\pi}(\varphi(\rho_{ij})) \| = \bar{p}(\varphi(\rho_{ij}))$. Hence, the $*$-representation $\pi = \bar{\pi} \circ \varphi$ of A in $\mathscr{H}$ satisfies $\pi(\rho_{ij}) = p(\rho_{ij})$. Now apply Lemma 4.2. This concludes the proof of our assertion.

THEOREM 4.4. *Let A be a CQG algebra and let $\mathfrak{P}$ denote the set of C^*-seminorms on A. The set $\mathfrak{P}$ is nonempty. For any $a \in A$, the number*

$$\| a \|_\infty = \sup_{p \in \mathfrak{P}} p(a) \tag{4.1}$$

is finite. The mapping $a \to \| a \|_\infty$ is a C^*-norm. The norm completion $A^\dagger$ of A with respect to $\| \cdot \|_\infty$ naturally is a unital C^*-algebra.

Clearly, the mapping $x \mapsto |\varepsilon(x)|$ of A into $\mathbb{R}$ is a C^*-seminorm. The fact that (4.1) is finite follows from Lemma 4.3. It then is clear that $\| \cdot \|_\infty$ is a C^*-seminorm. To show that $\| \cdot \|_\infty$ actually is a norm, it suffices to exhibit a C^*-norm on A. Let h denote the Haar functional on A. We define an inner product on A by putting $\langle a, b \rangle_h = h(b^*a)$. It follows from Proposition 3.2 and Theorem 3.7 that all the inner product axioms are satisfied. Left multiplication on A defines an algebra homomorphism of A into the algebra of linear operators on A such that the properties of Lemma 4.2 are satisfied. Hence, by Lemma 4.2, this algebra homomorphism can be extended to a $*$-representation π of A on the Hilbert space completion $\mathscr{H}_h$ of A by Lemma 4.2. Clearly, π is faithful and therefore $a \to \| \pi(a) \|_h$ is a C^*-norm. Here $\| \cdot \|_h$ denotes the operator norm on the space of bounded operators on the Hilbert space $\mathscr{H}_h$.

Remark 4.5. The essence of Theorem 4.4 can be extracted from [17] (after Proposition 3.5). N. Andruskiewitsch communicated to us that a detailed proof of

Theorem 4.4 is contained in a letter of his to A. Guichardet dated June 1993. This proof is also included in [7].

For obvious reasons, the norm $\|\cdot\|_\infty$ is called the *largest C^*-seminorm* on A. We call $A^\dagger$ the *universal C^*-algebra completion* of A. It is uniquely determined (up to a unique isomorphism) by the following universal property:

THEOREM 4.6. *Let A be a CQG algebra and let $\iota: A \to A^\dagger$ denote the canonical injection of A into its universal C^*-algebra completion $A^\dagger$. If B is a C^*-algebra and $\pi: A \to B$ a $*$-algebra homomorphism, then there exists a unique C^*-algebra homomorphism $\pi^\dagger: A^\dagger \to B$ with the property that $\pi^\dagger \circ \iota = \pi$.*

This is a direct consequence of the definition of $A^\dagger$ (cf. Theorem 4.4) and the fact that $a \mapsto \|\pi(a)\|$ is a C^*-seminorm.

PROPOSITION 4.7. *Let A be a CQG algebra. For any $a \in A$, one has $\|a\|_\infty = \sup_\pi \|\pi(a)\|$, where π runs through a complete set of irreducible $*$-representations of A.*

This follows by applying [4.1] to the C^*-algebra completion $A^\dagger$ of A.

Remark 4.8. Let A be a CQG algebra. Counit and comultiplication on A have unique extensions to $A^\dagger$. For the counit this follows from Theorem 4.4, since $\varepsilon: A \to \mathbb{C}$ is a one-dimensional $*$-representation of A. For the extension of Δ to $A^\dagger$, we need a suitable C^*-norm on the algebraic tensor product $A^\dagger \otimes A^\dagger$. We define the *injective cross norm* on $A^\dagger \otimes A^\dagger$ by setting

$$\|a\|_i = \sup_{\pi_1, \pi_2} \|(\pi_1 \otimes \pi_2)(a)\|, \quad a \in A^\dagger \otimes A^\dagger, \tag{4.2}$$

where π_1 and π_2 run through the set of $*$-representations of the C^*-algebra $A^\dagger$. The mapping $a \mapsto \|a\|_i$ clearly is a C^*-norm on $A^\dagger \otimes A^\dagger$. Now $a \mapsto (\pi_1 \otimes \pi_2)(\Delta(a))$ is a $*$-representation of A for any two $*$-representations π_1 and π_2 of A, so $\|(\pi_1 \otimes \pi_2)(\Delta(a))\| \leqslant \|a\|_\infty$, whence $\|\Delta(a)\|_i \leqslant \|a\|_\infty$. This implies that Δ extends to a continuous mapping of $A^\dagger$ into the completion of $A^\dagger \otimes A^\dagger$ with respect to $\|\cdot\|_i$.

Remark 4.9. Let A be a commutative CQG algebra. Then the irreducible $*$-representations of A are exactly its one-dimensional $*$-representations, in other words, the points of the compact group $G = G(A)$ corresponding to A. So $\|a\| = \sup_{x \in G} \|a(x)\|$, where we view $a \in A$ as a representative function on G. By the Peter–Weyl theorem, $A^\dagger$ is isometrically isomorphic to the C^*-algebra of continuous functions on the group G.

5. Comparison with Other Literature

(a) *Woronowicz* [16–18]

Woronowicz, in his influential 1987 paper [16], gives the following definition of a *compact matrix quantum group* (originally called *compact matrix pseudogroup*). It is a pair (B, u), where B is a unital C^*-algebra and $u = (u_{ij})_{i,j=1,\ldots,N}$ is an $N \times N$ matrix with entries in B, such that the following properties hold.

(1) The unital $*$-subalgebra A of B generated by the entries of u is dense in B.
(2) There exists a (necessarily unique) C^*-homomorphism $\Delta: B \to B \otimes B$ such that $\Delta(u_{ij}) = \sum_{k=1}^{N} u_{ik} \otimes u_{kj}$.
(3) There exists a (necessarily unique) linear anti-multiplicative mapping $S: A \to A$ such that $S \circ * \circ S \circ * = \mathrm{id}$ on A and $\sum_{k=1}^{N} S(u_{ik}) u_{kj} = \delta_{ij} 1 = \sum_{k=1}^{N} u_{ik} S(u_{kj})$.

In his note [18], Woronowicz shows that, instead of property (3), we may equivalently require:

(3′) The matrix u and its transpose are invertible.

Woronowicz now essentially shows (cf. [16] Prop. 1.8) that there exists a (necessarily unique) $*$-homomorphism $\varepsilon: A \to \mathbb{C}$ such that $\varepsilon(u_{ij}) = \delta_{ij}$ and that A becomes a Hopf $*$-algebra with comultiplication Δ, counit ε and antipode S. In [16], the notation $A, \mathscr{A}, \Phi, e, \kappa$ is used instead of our $B, A, \Delta, \varepsilon, S$, respectively. Note that the above $*$-algebra A is very close to what we have defined as a CMQG algebra (cf. Definition 2.3). However, it is not postulated and not yet obvious in the beginning of [16] that the corepresentations u and u' are unitarizable.

A central result in the paper (see [16], Theorem 4.2) is the existence of a *state* (normalized positive linear functional) h on the C^*-algebra B such that $(h \otimes \mathrm{id}) \circ \Delta(a) = h(a)1 = (\mathrm{id} \otimes h) \circ \Delta(a)$ for all $a \in B$. This state is necessarily unique and it is faithful on A. Then h may be called the Haar functional.

Woronowicz [16], Section 2, defines a *representation* of the compact matrix quantum group (B, u) on a finite-dimensional vector space V as a linear mapping $t: V \to V \otimes B$ such that $(t \otimes \mathrm{id}) \circ t = (\mathrm{id} \otimes \Delta) \circ t$. If $t(v) = 0$ implies $v = 0$, then the representation is called *nondegenerate* and if $t(V) \subset V \otimes A$, then the representation is called *smooth*. A smooth representation is nondegenerate iff $(\mathrm{id} \otimes \varepsilon) \circ t = \mathrm{id}$. Thus, corepresentations of A on finite-dimensional vector spaces, as defined in Section 1, correspond to nondegenerate smooth representations of (B, u) in [16].

As a consequence of the existence of the Haar functional, it is shown in [16], Theorem 5.2, Proposition 3.2, that nondegenerate smooth representations of (B, u) are unitarizable. This implies that the dense $*$-algebra A of B is a CMQG algebra.

Conversely, if we start with a CMQG algebra A with fundamental corepresentation u as in Proposition 2.3, then we have shown the existence of a positive definite Haar functional h on A (cf. Theorem 3.7) without using C^*-algebras, and we have next obtained a C^*-completion $A^\dagger$ of A by making essential use of the existence of a positive definite Haar functional (cf. Section 4). Then it is clear that the pair $(A^\dagger, u)$ is a compact matrix quantum group in the sense of Woronowicz. However, the C^*-algebra $A^\dagger$ possesses the universal property of Theorem 4.6 but this is not necessarily the case with the compact matrix quantum groups (B, u) of Woronowicz, since the norm induced by B on A may not be the largest C^*-seminorm on A.

Accordingly, the counit $\varepsilon\colon \mathcal{A} \to \mathbb{C}$ does not necessarily have a continuous extension to a linear functional on A (cf. [16] (second Remark to Proposition 1.8)).

In his paper [17], Woronowicz starts with a compact matrix quantum group (B, u) in the sense of [16], then constructs out of its finite-dimensional unitary representations a so-called complete concrete monoidal W^*-category (see [17], Theorem 1.2) and next constructs from any such category a compact matrix quantum group $(A^\dagger, u)$. Then $A^\dagger$ is not necessarily isomorphic to A, but it has the universal property of Theorem 4.6 with respect to the CMQG algebra A generated by the entries of u. The relation between CQG algebras and monoidal W^*-categories is much closer (cf. [4]).

Both in [16] and in this Letter, there is a similar key result [16] (last statement of Theorem 5.4, resp. Proposition 3.6). We got the idea of the statement and proof of Proposition 3.6 from [16], but in this Letter, different from [16], the positivity and faithfulness of the Haar functional on $\mathcal{A}$ is a corollary rather than a prerequisite.

(b) *Woronowicz [19] and S. Wang [14, 15]*

Woronowicz [19] defines a *compact quantum group* as a pair (B, Δ), where B is a separable unital C^*-algebra and $\Delta\colon B \to B \otimes B$ is a C^*-homomorphism, such that the following properties hold.

(1) $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$.

(2) $\mathrm{Span}\{(b \otimes 1)\Delta(c) \mid b, c \in B\}$ and $\mathrm{Span}\{(1 \otimes b)\Delta(c) \mid b, c \in B\}$ are dense subspaces of $B \otimes B$.

In particular, if (B, u) is a compact matrix pseudogroup as defined in [16] and if Δ is the corresponding comultiplication, then (B, Δ) is a compact quantum group as just defined. Conversely, it is shown in [19] that, if (B, Δ) is a compact quantum group and if A is the set of all linear combinations of matrix elements of finite-dimensional unitary representations of (B, Δ), then A is a dense $*$-subalgebra of B and A is a Hopf $*$-algebra. The existence of a Haar functional is also shown. It is observed that the representation theory as developed in [16] can be formulated in a similar way for compact quantum groups.

It is pointed out by Wang ([14], Remark 2.2) that the results of [19] remain true if separability of the C^*-algebra B is no longer required, but if it is assumed instead that the C^*-algebra B has a faithful state. This observation would imply that a compact quantum group (B, Δ) in the sense of Wang gives rise to a CQG algebra A (A being constructed from B as in the previous paragraph), and that conversely each CQG algebra A would give rise to a compact quantum group (B, Δ) (B being completion of A with respect to maximal C^*-seminorm), provided B has a faithful state.

In [15], Wang defines the notion of (noncommutative) Krein algebra, which is essentially equivalent to our notion of CQG algebra.

(c) *Effros and Ruan* [5]

In different terminology, CQG algebras were earlier introduced by Effros and Ruan [5]. They defined these algebras as cosemisimple Hopf algebras with a so-called standard $*$-operation and they called these structures *discrete quantum groups*. This name was motivated by the fact that special examples of these algebras are provided by the group algebra of a discrete group, while the name CQG algebra comes from the class of examples where we deal with the algebra of representative functions on a compact group. In the final section of [5], the authors define a *compact quantum group* as a natural generalization of the compact matrix quantum groups defined in [16]. Their definition involves a unital C^*-algebra B with a dense unital $*$-subalgebra A, where A is a CQG algebra (in the terminology of the present paper) and the comultiplication on A extends continuously to B. Conversely, they show that a CQG algebra A gives rise to a compact quantum group according to their definition. This involves a C^*-completion, for which a Haar functional h on A is needed. This Haar functional is obtained in a way very different from the method in the present paper. The authors first show the existence of a left Haar functional φ on a certain subspace of the linear dual of A. Then h is constructed in terms of φ. For a detailed comparison of [5] with the results in this Letter, see [9] (section 6).

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